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COMMENT

On Kronecker products of irreducible representations of the symmetric group†

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Abstract. A formula for the reduction of Kronecker products of irreducible representations of the symmetric group due to Littlewood is derived using the algebra of S functions.

The irreducible characters of S_n, the symmetric group of permutations of n objects, may be denoted by $\chi(\lambda)$, where $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of n, i.e. a weakly decreasing sequence of non-negative integers whose sum is n. Given two irreducible characters $\chi(\lambda)$ and $\chi(\mu)$ the reduction of their product into its irreducible components,

$$\chi(\lambda)\chi(\mu)=\sum_{\nu} g_{\lambda\mu\nu}\chi(\nu) \tag{1}$$

may, in principle, be performed easily using character tables. In practice, however, this method rapidly becomes unwieldy as n increases. This problem has stimulated the search for more efficient ways of performing these products. The most remarkable results in this direction were obtained by Murnaghan, who, in a series of papers (Murnaghan 1951, 1955a, b, c 1956) produced formulae for characters, Kronecker products and symmetrised Kronecker products which may be applied to all values of n. (A useful commentary on these results may be found in Gorenstein (1974).)

The key idea introduced in these papers is the notion of a reduced partition $\lambda=(n-|\sigma|, \sigma)$, where σ is a partition and n a fixed positive integer. Thus λ is a sequence of integers whose first term is $n-|\sigma|$, the second σ_1, the third σ_2 and so on. To each reduced partition $\lambda=(n-|\sigma|, \sigma)$ there corresponds an element of the character ring of S_n denoted by $\hat{\chi}(\sigma)$, which is, up to a sign factor, an irreducible character or zero. If $n-|\sigma|\geqslant\sigma_1$, i.e. if n is sufficiently large, then λ is a partition and $\hat{\chi}(\sigma)=\chi(\lambda)$. On the other hand, if $n-|\sigma|<\sigma_1$ then the character corresponding to $\hat{\chi}(\sigma)$ must be calculated using modification rules (Littlewood 1950, p 98). Equivalently $\hat{\chi}(\sigma)$ may be defined by means of a determinant depending on λ (see below).

It is a remarkable fact that the products of the reduced characters $\hat{\chi}(\sigma)$ do not depend explicitly on the value of n, except through modification rules which need only

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be applied for small values of n. For example, Hamermesh (1962) records the following:

$$\begin{aligned}
&\hat{\chi}(1)\hat{\chi}(1)=\hat{\chi}(2)+\hat{\chi}(1^2)+\hat{\chi}(1)+\hat{\chi}(0)\\
&\hat{\chi}(1)\hat{\chi}(2)=\hat{\chi}(3)+\hat{\chi}(2,1)+\hat{\chi}(2)+\hat{\chi}(1^2)+\hat{\chi}(1)\\
&\hat{\chi}(1)\hat{\chi}(1^2)=\hat{\chi}(2,1)+\hat{\chi}(1^3)+\hat{\chi}(2)+\hat{\chi}(1^2)+\hat{\chi}(1)\\
&\hat{\chi}(2)\hat{\chi}(2)=\hat{\chi}(4)+\hat{\chi}(3,1)+\hat{\chi}(2^2)+\hat{\chi}(3)+2\hat{\chi}(2,1)+\hat{\chi}(1^3)+2\hat{\chi}(2)+\hat{\chi}(1^2)+\hat{\chi}(1)+\hat{\chi}(0).
\end{aligned}\tag{2}$$

More results of this type can be found in Murnaghan's papers.

Although it is quite possible to develop the whole theory of these reduced characters solely from the point of view of the symmetric group, it turns out to be somewhat simpler (not least from a notational point of view) to make use of the $\mathbb{Z}$-linear mapping

$$\vartheta:\mathfrak{R}\rightarrow\Lambda \qquad \vartheta(\chi(\lambda))=\{\lambda\}$$

between the ring $\mathfrak{R}$, consisting of the direct sum of character rings of the symmetric groups, and the ring Λ, of symmetric functions (see, for example, Macdonald 1979). Here $\{\lambda\}$ is the S function labelled by the partition $\lambda (S_\lambda$ in Macdonald's notation). The product (1) gives rise in Λ to the 'inner product', $\{\lambda\}\circ\{\mu\}$, of S functions, i.e.

$$\vartheta(\chi(\lambda)\chi(\mu))=\{\lambda\}\circ\{\mu\}. \tag{3}$$

While the outer or 'Littlewood–Richardson' product of S functions, $\{\lambda\}\{\mu\}$, corresponds to the character $\mathrm{ind}_{m,n}(\chi(\lambda)\chi(\mu))$ of S_{m+n} induced from the character $\chi(\lambda)\chi(\mu)$ of $S_m\times S_n$, i.e.

$$\vartheta(\mathrm{ind}_{m,n}(\chi(\lambda)\chi(\mu)))=\{\lambda\}\{\mu\}. \tag{4}$$

Thus we may perform our calculations in Λ and then interpret the results in terms of symmetric group characters.

We should introduce at this point the idea of S-function division which is defined via the Littlewood–Richardson product as follows. First write

$$\{\lambda\}\{\mu\}=\sum_\nu \Gamma^\nu_{\lambda\mu}\{\nu\} \tag{5}$$

where $\Gamma^\nu_{\lambda\mu}$ are non-negative integers known as the Littlewood–Richardson coefficients (Littlewood and Richardson 1934). Using these we may define a bilinear product

$$\{\lambda\}/\{\mu\}=\sum_\nu \Gamma^\lambda_{\mu\nu}\{\nu\} \tag{6}$$

which is known as S function division. At first sight there seems to be little to be gained from this definition since it might be argued that any formula involving $\{\lambda\}\{\mu\}$, $\{\lambda\}/\{\mu\}$ or $\{\lambda\}\circ\{\mu\}$ ultimately boils down to a statement concerning Littlewood–Richardson coefficients and the coefficients $g_{\lambda\mu\nu}$ (which can in fact be expressed in terms of Littlewood–Richardson coefficients). This argument can, however, be countered by citing several advantages of this new notation. Firstly, formulae involving S-function operations can be given a transparent interpretation in terms of tensor methods which facilitates both their manipulation and discovery (in fact it was using just this line of reasoning that led Littlewood to his product formula that will soon be described). Secondly, there are several identities satisfied by the Littlewood–Richardson coefficients that are most simply expressed and manipulated as identities satisfied by the three operations of inner and outer multiplication and division. These will play a key role later in the proof presented below.

Now let $\langle\sigma\rangle=\vartheta(\hat{\chi}(\sigma))$. Littlewood (1958) has stated the following:

$$\langle\sigma\rangle\circ\langle\tau\rangle=\sum_{\delta,\rho,\beta}\langle\{\sigma/\delta\beta\}\{\tau/\rho\beta\}\{\delta\}\circ\{\rho\}\rangle \tag{7}$$

where the sum is over all partitions δ, ρ and β. The advantage of this result is that it allows the calculation of general products of the form (2), provided we know certain particular inner products $\{\delta\}\circ\{\rho\}$ and have a method of calculating the various Littlewood–Richardson coefficients (for which there exists a simple combinatorial procedure). An example will make this process clearer. Consider the product $\langle 1\rangle\circ\langle 1\rangle$ ($\hat{\chi}(1)\hat{\chi}(1)$ in terms of symmetric group characters); in this case (7) yields after a little simplification

$$\langle 1\rangle\circ\langle 1\rangle=\langle\{1\}\{1\}(\{0\}\circ\{0\})\rangle+\langle\{0\}\circ\{0\}\rangle+\langle\{1\}\circ\{1\}\rangle$$

and since $\{0\}\circ\{0\}=\{0\}$, $\{1\}\circ\{1\}=\{1\}$ and $\{1\}\{1\}=\{2\}+\{1^2\}$ we obtain

$$\langle 1\rangle\circ\langle 1\rangle=\langle 2\rangle+\langle 1^2\rangle+\langle 1\rangle+\langle 0\rangle$$

as expected. This result now applies to any value of n, for example for $n=10$,

$$\{9,1\}\circ\{9,1\}=\{8,2\}+\{8,1^2\}+\{9,1\}+\{10\}$$

corresponding to

$$\chi(9,1)\chi(9,1)=\chi(8,2)+\chi(8,1^2)+\chi(9,1)+\chi(10)$$

in S_{10}.

Littlewood sketched a proof of (7) making use of the fact that S_n may be characterised as the subgroup of $\mathrm{SU}(n-1)$ leaving invariant certain symmetric tensors of second and third rank. It is the aim of this comment to provide a proof of this result which uses only the properties of S functions. The notation used follows that of Littlewood (1950, 1958) (see also Macdonald (1979)). Note that $\{\sigma\circ\tau\}$, $\{\sigma/\tau\}$ and $\{\sigma\tau\}$ are used as abbreviations for $\{\sigma\}\circ\{\tau\}$, $\{\sigma\}/\{\tau\}$ and $\{\sigma\}\{\tau\}$, respectively. Throughout sums not shown explicitly are over all partitions in the case of greek letters and from zero to infinity in the case of latin letters.

An advantage of working with symmetric functions is that the determinantal expansion,

$$\{\lambda\}=\det\{\lambda_i-i+j\} \tag{8}$$

of $\{\lambda\}$ in terms of the complete symmetric functions $\{m\}$ gives rise to a natural definition of $\langle\sigma\rangle$ as a determinant. Thus if $\lambda=(n-|\sigma|,\sigma)$, as above, then we set

$$\langle\sigma\rangle=\det\{\lambda_i-i+j\}. \tag{9}$$

If $n-|\sigma|\geqslant\sigma_1$ then λ is a partition and so, comparing (8) and (9), we have $\langle\sigma\rangle=\{\lambda\}$. On the other hand, if $n-|\sigma|<\sigma_1$ then by rearranging the rows of the determinant in equation (9) we obtain $\pm$ the determinantal expansion of some S function $\{\tau\}$, or zero. This is the modification rule referred to above. We regard $\hat{\chi}(\sigma)$ as being defined by (9) via ϑ^{-1}.

By expanding the first row of $\det\{\lambda_i-i+j\}$ another form of $\langle\sigma\rangle$ is obtained,

$$\langle\sigma\rangle=\sum_{a=0}^{l(\sigma)}(-1)^a\{n-s+a\}\{\sigma/1^a\} \tag{10}$$

where $s=|\sigma|$ and $l(\sigma)$ is the depth (number of parts) of σ.

To proceed, we require a formula relating S-function division and multiplication due to Foulkes (1949) and a formula that relates the inner and outer products of S functions. The former takes the form

$$(\{\lambda\}\{\mu\})/\{\nu\}=\sum_{\rho}(\{\lambda\}/\{\rho\})(\{\mu\}/\{\nu/\rho\}) \tag{11}$$

and the latter has been described by Littlewood (1956),

$$(\{\lambda\}\{\tau\})\circ\{\sigma\}=\sum_{\delta}(\{\lambda\}\circ\{\delta\})(\{\tau\}\circ\{\sigma/\delta\}). \tag{12}$$

As noted above, these two formulae owe their origin to identities satisfied by the Littlewood–Richardson coefficients and the coefficients $g_{\lambda\mu\nu}$. These may be found in a straightforward manner by using equations (1) (or rather its equivalent form for S functions), (5) and (6) to express the various products in terms of their structure constants and then equating the coefficients of the basis $\{\nu\}$. Somewhat surprisingly the resulting expression in the case of equation (12) may be shown after a little manipulation, the details of which may safely be left to the reader, to be equivalent to the identity

$$(\{\lambda\}\circ\{\tau\})/\{\sigma\}=\sum_{\delta}(\{\lambda\}/\{\sigma\circ\delta\})\circ(\{\tau/\delta\}) \tag{13}$$

which appears to be quite different from (12). We shall make use of this relationship later.

It should be noted that the proof of the product formula (7) which follows hinges on the three equations (8), (11) and (12) and the results that will be derived from them. Since proofs of each of these exist in the literature or follow immediately from known results, it would be inappropriate to give detailed derivations here. We simply note that (8) is contained in Macdonald (1979), (11) was derived by Foulkes (1949) in a slightly different form and (12) follows from the expansion of the product $\{\lambda;\boldsymbol{xy}\}\{\mu;\boldsymbol{xy}\}$ in two different ways, where $(\boldsymbol{xy})$ is the set of variables $(x_1y_1, x_1y_2, \ldots, x_my_n)$ (see Littlewood (1956) theorem III, for a proof based directly on properties of the symmetric group). In this last case use must be made of the standard expansion

$$\{\lambda;\boldsymbol{xy}\}=\sum_{\sigma}\{\lambda\circ\sigma;\boldsymbol{x}\}\{\sigma;\boldsymbol{y}\} \tag{14}$$

a proof of which may once again be found in Macdonald (1979).

Let us now commence the proof of equation (7). We shall first require a rather complicated expansion that follows easily, but somewhat tediously, from (11) and (12). Consider $(\{\alpha\}\{\beta\})\circ(\{\sigma\}(\{\tau\})$; using (12) we have

$$(\{\alpha\}\{\beta\})\circ(\{\sigma\}\{\tau\})=\sum_{\delta}\{\alpha\circ\delta\}(\{\beta\}\circ(\{\sigma\tau\}/\{\delta\}))$$

and applying (11) to the second term and changing the summation variables (the odd labelling of these variables is to simplify the final expression),

$$=\sum_{\gamma_2,\delta_1}(\{\alpha\}\circ\{\gamma_2\delta_1\})(\{\beta\}\circ(\{\sigma/\gamma_2\}\{\tau/\delta_1\})).$$

Once again using (12) we find

$$=\sum_{\substack{\gamma_1,\gamma_2\\ \delta_1}}(\{\alpha/\gamma_1\}\circ\{\gamma_2\})\{\gamma_1\circ\delta_1\}(\{\beta\}\circ(\{\sigma/\gamma_2\}\{\tau/\delta_1\})).$$

Changing the summation variables yields

$$=\sum_{\substack{\gamma_1,\gamma_2\\ \delta_1}}(\{\alpha/\gamma_1\}\circ\{\sigma/\gamma_2\})(\{\gamma_1\}\circ\{\tau/\delta_1\})(\{\beta\}\circ\{\gamma_2\delta_1\})$$

and finally applying (12) to the last term

$$=\sum_{\substack{\gamma_1,\gamma_2\\ \delta_1,\delta_2}}(\{\alpha/\gamma_1\}\circ\{\sigma/\gamma_2\})(\{\gamma_1\}\circ\{\tau/\delta_1\})(\{\beta/\delta_2\}\circ\{\gamma_2\})\{\delta_2\circ\delta_1\}.$$

Putting this all together we have the following result:

$$(\{\alpha\}\{\beta\})\circ(\{\sigma\}\{\tau\})=\sum_{\delta_1\delta_2\gamma_1\gamma_2}(\{\alpha/\gamma_1\}\circ\{\sigma/\gamma_2\})(\gamma_1\circ\{\tau/\delta_1\})(\gamma_2\circ\{\beta/\delta_2\})\{\delta_1\circ\delta_2\}. \qquad (15)$$

Now consider the product $\langle\sigma\rangle\circ\langle\tau\rangle$, using (10) and (15)

$$\langle\sigma\rangle\circ\langle\tau\rangle=\sum_{a,b,g_1,g_2,\delta_1,\delta_2}(-1)^{a+b}\{n-s+a-g_1\}\circ\{n-t+b-g_2\}$$
$$\times\{g_1\}\circ\{\tau/1^b\delta_1\}\{g_2\}\circ\{\sigma/1^a\delta_2\}\{\delta_1\}\circ\{\delta_2\} \qquad (16)$$

where $s=|\sigma|$ and $t=|\tau|$. From the second term $g_1=t-b-\delta$ and from the third $g_2=s-a-\delta(\delta=|\delta_1|=|\delta_2|)$. Hence (16) simplifies to

$$\langle\sigma\rangle\circ\langle\tau\rangle=\sum_{a,b}\sum_{\delta_1,\delta_2}(-1)^{a+b}\{n-s-t+a+b+\delta\}\{\tau/1^b\delta_1\}\{\sigma/1^a\delta_2\}\{\delta_1\}\circ\{\delta_2\}. \qquad (17)$$

Now we have the simple result (Littlewood 1950, p 107)

$$\sum_{\alpha}\{\alpha\}\{\beta/\alpha^*\}=\begin{cases}\{0\} & \text{if } \{\beta\}=\{0\}\\ \text{zero} & \text{otherwise}\end{cases} \qquad (18)$$

where $\{\alpha\}=(-1)^{|\alpha|}\{\alpha'\}$, α' being the partition conjugate to α. Using (18) the following insertion may be made into (17):

$$\langle\sigma\rangle\circ\langle\tau\rangle=\sum_{a,b}\sum_{\delta_1,\delta_2}\sum_{\alpha,\beta}(-1)^{a+b}\{n-s-t+a+b+\delta+2|\beta|\}\{\tau/1^b\delta_1\alpha(\beta/\alpha^*)\}$$
$$\times\{\sigma/1^a\delta_2\beta\}\{\delta_1\}\circ\{\delta_2\} \qquad (19)$$

and rearranging the sum over α and β yields

$$\langle\sigma\rangle\circ\langle\tau\rangle=\sum_{a,b}\sum_{\delta_1,\delta_2}\sum_{\alpha,\beta}(-1)^{a+b}\{n-s-t+a+b+\delta+2|\beta|+2|\alpha|\}\{\tau/1^b\delta_1\alpha\beta\}$$
$$\times\{\sigma/1^a\delta_2\alpha^*\beta\}\{\delta_1\}\circ\{\delta_2\}. \qquad (20)$$

Next, rearranging the sums over δ_1, δ_2 and α gives

$$\langle\sigma\rangle\circ\langle\tau\rangle=\sum_{a,b}\sum_{\delta_1,\delta_2}\sum_{\alpha,\beta}(-1)^{a+b}\{n-s-t+a+b+\delta+2|\beta|+|\alpha|\}\{\tau/1^b\delta_1\beta\}$$
$$\times\{\sigma/1^a\delta_2\beta\}\{\delta_1/\alpha\}\circ\{\delta_2/\alpha^*\}. \qquad (21)$$

Using equation (13) and the fact that $\{1^{|\lambda|}\}\circ\{\lambda\}=\{\lambda'\}$ it is easy to see that

$$\sum_k(-1)^k(\{\mu\}\circ\{\lambda\})/\{1^k\}=\sum_\alpha\{\mu/\alpha\}\circ\{\lambda/\alpha^*\}. \qquad (22)$$

Thus equation (21) becomes

$$\langle\sigma\rangle\circ\langle\tau\rangle=\sum_{a,b,c}\sum_{\delta_1,\delta_2}\sum_{\beta}(-1)^{a+b+c}\{n-s-t+a+b+c+\delta+2|\beta|\}\{\tau/1^b\delta_1\beta\}$$
$$\times\{\sigma/1^a\delta_2\beta\}(\{\delta_1\}\circ\{\delta_2\})/\{1^c\} \qquad (23)$$

and, from (11) and some well known properties of Littlewood-Richardson coefficients,

$$\sum_m(-1)^m(\{\alpha\}\{\beta\}\{\gamma\})/\{1^m\}=\sum_{a,b,c}(-1)^{a+b+c}\{\alpha/1^a\}\{\beta/1^b\}\{\gamma/1^c\}. \qquad (24)$$

It follows that we have

$$\langle\sigma\rangle\circ\langle\tau\rangle=\sum_m\sum_{\delta_1,\delta_2}\sum_\beta(-1)^m\{n-s-t+\delta+2|\beta|+m\}$$
$$\times(\{\tau/\delta_1\beta\})\{\sigma/\delta_2\beta\}\{\delta_1\}\circ\{\delta_2\})/\{1^m\}. \qquad (25)$$

Thus from equation (10)

$$\langle\sigma\rangle\circ\langle\tau\rangle=\sum_{\delta,\rho,\beta}\langle\{\sigma/\delta\beta\}\{\tau/\rho\beta\}\{\delta\}\circ\{\rho\}\rangle \tag{26}$$

which is equation (7) as required.

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